International Journal of Number Theory
Vol. 12, No. 6 (2016) 1497–1507

DOI: 10.1142/S1793042116500925

On the cardinality of β-expansions of some numbers

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Received 21 May 2015
Accepted 27 August 2015
Published 28 October 2015

Let $\beta > 1$. It is well known that every $x \in [0, \lfloor\beta\rfloor/(\beta - 1)]$ has a β-expansion of the form $x = \sum_{k=1}^{\infty} \delta_k \beta^{-k}$ with $\delta_i \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$, where $\lfloor\beta\rfloor$ denotes the largest integer not exceeding β. Let $\Sigma_\beta(x)$ and $\Sigma_{\beta,n}(x)$ denote the sets of all β-expansions of x and the set of n-prefixes of all β-expansions of x, respectively. We show that $\#\Sigma_\beta(x) = 2^{\aleph_0}$, $\dim_H \Sigma_\beta(x) > 0$ and $\lim_{n\to\infty} \frac{1}{n} \log \#\Sigma_{\beta,n}(x) > 0$ are equivalent under a certain finiteness condition.

Keywords: β-Expansion; Hausdorff dimension; directed graph.

Mathematics Subject Classification 2010: 11A63, 28A80

1. Introduction

Let $\beta > 1$ be a non-integer. We consider expansions of $x \in J_\beta := [0, \lfloor\beta\rfloor/(\beta - 1)]$ of the form

$$x = \sum_{i=1}^{\infty} \frac{\delta_i}{\beta^i} \quad \text{with } \delta_i \in \{0, 1, \ldots, \lfloor\beta\rfloor\},$$

where $\lfloor\beta\rfloor$ denotes the largest integer not exceeding β. The infinite sequence $(\delta_i)_{i=1}^{\infty}$ is called a β-expansion of x. We write (δ_i) instead of $(\delta_i)_{i=1}^{\infty}$ for simplicity, except when we want to emphasize the first digit of (δ_i). The study of expansions in non-integer bases were pioneered by the papers of Rényi [20] and Parry [19]. Let $\Sigma_\beta(x)$ denote the set of all β-expansions of x and $\Sigma_{\beta,n}(x)$ the set of n-prefixes of all β-expansions of x, i.e.

$$\Sigma_\beta(x) = \left\{ (\delta_i) \in \{0, 1, \ldots, \lfloor\beta\rfloor\}^{\mathbb{N}} : x = \sum_{i=1}^{\infty} \delta_i \beta^{-i} \right\},$$

$$\Sigma_{\beta,n}(x) = \{(\varepsilon_i)_{i=1}^{n} \in \{0, 1, \ldots, \lfloor\beta\rfloor\}^n : \text{ there exists } (\varepsilon_i)_{i=n+1}^{\infty} \in \{0, 1, \ldots, \lfloor\beta\rfloor\}^{\mathbb{N}} \text{ such that } (\varepsilon_i) \in \Sigma_\beta(x)\}.$$

The set $\Sigma_\beta(x)$ plays an important role in the investigation of representations for real numbers in non-integer bases. In the past two decades the set $\mathcal{U}$ of all such $\beta > 1$ for which $\#\Sigma_\beta(1) = 1$ has been widely investigated and numerous interesting results have been obtained (see [1, 7–10, 13–15] and references therein). Here and hereafter $\#A$ denotes the cardinality of a set A. Recently, the Hausdorff dimension of the set of all points belonging to J_β which have a unique β-expansion was calculated in [16, 17, 24].

On the other hand, the cardinality of the set $\Sigma_\beta(x)$ also has received a lot of attention. Glendinning and Sidorov [12] showed that the Komornik–Loreti constant (see [1, 13]) is the critical value which separates the cardinality of set $\Sigma_\beta(x)$ being uncountable from countable. It was shown in [10] that each $x \in J_\beta$ has $2^{\aleph_0}$ different β-expansions if $\beta \in (1, (1+\sqrt{5})/2)$. This result was strengthened in [6, 21, 22] to get that for any non-integer $\beta > 1$, almost every $x \in J_\beta$ has $2^{\aleph_0}$ distinct β-expansions. Moveover, some similar results also hold in two dimensions [23]. Recently, Feng and Sidorov [11] showed that for any Pisot number $\beta > 1$ there has $\lim_{n\to\infty} \frac{1}{n} \log \#\Sigma_{\beta,n}(x) > 0$ for almost every $x \in J_\beta$. Here $\lim_{n\to\infty} \frac{1}{n} \log \#\Sigma_{\beta,n}(x)$ is called the growth rate of the set $\Sigma_{\beta,n}(x)$, provided that the limit exists. The growth rate was further investigated by Baker [2, 3]. In [3] Baker showed that under some conditions the growth rate of the set $\Sigma_{\beta,n}(x)$ and Hausdorff dimension of the set $\Sigma_\beta(x)$ are equal and explicitly calculable. In fact, all the quantities $\#\Sigma_\beta(x)$, $\lim_{n\to\infty} \frac{1}{n} \log \#\Sigma_{\beta,n}(x)$ and $\dim_H \Sigma_\beta(x)$ reveal the complexity of the set $\Sigma_\beta(x)$. In this paper we are mainly concerned with the relation among them. We show that under a certain finiteness condition, the cases of $\#\Sigma_\beta(x) = 2^{\aleph_0}$, $\dim_H(\Sigma_\beta(x)) > 0$ and $\lim_{n\to\infty} \frac{1}{n} \log \#\Sigma_{\beta,n}(x) > 0$ are equivalent.

As one knows the set $J_\beta = [0, \lfloor\beta\rfloor/(\beta - 1)]$ can be regarded as the self-similar set generated by the iterated function system (IFS) $\{f_k(x) = \beta^{-1}(x + k) : k = 0, 1, \ldots, \lfloor\beta\rfloor\}$, i.e.

$$J_\beta = \bigcup_{k=0}^{\lfloor\beta\rfloor} f_k(J_\beta).$$

As usual, a coding mapping $\Pi : \{0, 1, \ldots, \lfloor\beta\rfloor\}^{\mathbb{N}} \to J_\beta$ is then defined by

$$\Pi((\delta_i)) = \sum_{i=1}^{\infty} \frac{\delta_i}{\beta^i} \quad \text{for } (\delta_i) \in \{0, 1, \ldots, \lfloor\beta\rfloor\}^{\mathbb{N}}.$$

Then we have that Π is surjective and for each $x \in J_\beta$ and $n \in \mathbb{N}$

$$\Sigma_\beta(x) = \Pi^{-1}(x) \quad \text{and} \quad \Sigma_{\beta,n}(x) = \{(\delta_i) \,|\, n : (\delta_i) \in \Pi^{-1}(x)\},$$

where $(\delta_i) \,|\, n = (\delta_i)_{i=1}^n$ is the n-prefix of (δ_i).

Denote $I_k = f_k(J_\beta) = [\frac{k}{\beta}, \frac{k}{\beta} + \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}]$ and partition the interval $J_\beta = [0, \lfloor\beta\rfloor/(\beta-1)]$ into *switch regions* S_i and *equality regions* E_i by letting

$$S_i = I_{i-1} \cap I_i = \left[\frac{i}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i-1}{\beta}\right] \neq \emptyset \quad \text{for } i = 1, 2, \ldots, \lfloor\beta\rfloor$$

and

$$E_i = I_i \setminus \bigcup_{k=1}^{\lfloor\beta\rfloor} S_k \quad \text{for } i = 0, 1, \ldots, \lfloor\beta\rfloor.$$

Thus we have

$$E_i = \begin{cases} \left[0, \dfrac{1}{\beta}\right) & i = 0, \\ \left(\dfrac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \dfrac{i-1}{\beta}, \dfrac{i+1}{\beta}\right) & i = 1, 2, \ldots, \lfloor\beta\rfloor - 1, \\ \left(\dfrac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \dfrac{\lfloor\beta\rfloor-1}{\beta}, \dfrac{\lfloor\beta\rfloor}{\beta-1}\right] & i = \lfloor\beta\rfloor. \end{cases}$$

So one has

$$J_\beta = E_0 \cup S_1 \cup E_1 \cup S_2 \cup E_2 \cup \cdots \cup S_{\lfloor\beta\rfloor} \cup E_{\lfloor\beta\rfloor},$$

where the union is disjoint and all intervals in the union are lined up in this order from left to right. In addition

$$I_k = S_k \cup E_k \cup S_{k+1} \quad \text{for } k = 0, 1, \ldots, \lfloor\beta\rfloor,$$

where we adopt the convention that $S_0 = S_{\lfloor\beta\rfloor+1} = \emptyset$. Let $S_\beta = \bigcup_{i=1}^{\lfloor\beta\rfloor} S_i$. Let

$$T_{\beta,k}(x) = \beta x - k \text{ with } k = 0, 1, \ldots, \lfloor\beta\rfloor.$$

Then $T_{\beta,k}(x) \in J_\beta$ if and only if $x \in I_k = S_k \cup E_k \cup S_{k+1}$. Note that

$$T_{\beta,\delta_1}\left(\sum_{i=1}^{\infty} \frac{\delta_i}{\beta^i}\right) = \sum_{i=2}^{\infty} \frac{\delta_i}{\beta^{i-1}} = \Pi((\delta_i)_{i=2}^{\infty}).$$

This implies the following facts (cf. [3, Lemma 1.1; 5, Theorem 2]):

(I) $(\delta_i)_{i=1}^\infty \in \Sigma_\beta(x)$ if and only if $T_{\beta,\delta_n} \circ \cdots \circ T_{\beta,\delta_1}(x) \in J_\beta$ for all $n \geq 1$.

(II) A finite block of sequence $(\delta_i)_{i=1}^n \in \{0, 1, \ldots, \lfloor\beta\rfloor\}^n$ appears in a β-expansion of x if and only if there exist finite digits $\tau_1, \ldots, \tau_k$ from $\{0, 1, \ldots, \lfloor\beta\rfloor\}$ such that $T_{\beta,\delta_n} \circ \cdots \circ T_{\beta,\delta_1} \circ T_{\beta,\tau_k} \circ \cdots \circ T_{\beta,\tau_1}(x) \in J_\beta$.

For $x \in J_\beta$ and $n \in \mathbb{N}$ denote

$$\hat{J}_{\beta,n}(x) = \{T_{\beta,\delta_n} \circ \cdots \circ T_{\beta,\delta_1}(x) : (\delta_i)_{i=1}^\infty \in \Sigma_\beta(x)\} = \left\{\sum_{k=1}^\infty \frac{\delta_{n+k}}{\beta^k} : (\delta_i)_{i=1}^\infty \in \Sigma_\beta(x)\right\}$$

and

$$\hat{J}_\beta(x) = \bigcup_{n=0}^\infty \hat{J}_{\beta,n}(x) \text{ with } \hat{J}_{\beta,0}(x) = \{x\}.$$

If we use σ to denote the left shift on $\{0, 1, \ldots, \lfloor\beta\rfloor\}^{\mathbb{N}}$, then

$$\hat{J}_\beta(x) = \bigcup_{n=0}^\infty \hat{J}_{\beta,n}(x) = \bigcup_{n=0}^\infty \Pi(\sigma^n \Sigma_\beta(x)).$$

Set

$$\hat{S}_\beta(x) = \hat{J}_\beta(x) \cap S_\beta.$$

For a finite sequence $\mathbf{e} \in \bigcup_{k=1}^\infty \mathbb{Z}^k$ we denote by $\mathbf{i}(\mathbf{e})$ its initial digit, and by $\mathbf{e}^\infty$ the infinite sequence obtained by concatenating $\mathbf{e}$ to itself infinite many times. An infinite sequence $(c_i)_{i=1}^\infty$ of integers is said eventually periodic if there exist $\mathbf{a} \in \bigcup_{k=0}^\infty \mathbb{Z}^k$ and $\mathbf{b} \in \bigcup_{k=1}^\infty \mathbb{Z}^k$ such that $(c_i)_{i=1}^\infty = \mathbf{a}\mathbf{b}^\infty$. Here $\mathbb{Z}^0$ consists of empty sequence.

If $\hat{J}_\beta(x) = \bigcup_{n=0}^\infty \hat{J}_{\beta,n}(x)$ is finite such that $\hat{S}_\beta(x) \neq \emptyset$, then β is an algebraic integer determined by some monic polynomial with integer coefficients. In fact, for this case one can take two distinct eventually periodic sequences (δ_i) and (ε_i) from $\Sigma_\beta(x)$ with n being the least number such that $\delta_j \neq \varepsilon_j$ then $|\delta_n - \varepsilon_n| = 1$. Thus the following equality leads to such a monic polynomial:

$$\sum_{i=1}^\infty \frac{\delta_i}{\beta^i} = \sum_{i=1}^\infty \frac{\varepsilon_i}{\beta^i}.$$

As to the finiteness of $\hat{J}_\beta(x)$, Bogmér *et al.* showed in [4] that $\hat{J}_\beta(1)$ is finite if β is a Pisot number. Recently, Baker [3] generalized their result and showed for Pisot number β, $\hat{J}_\beta(x)$ is finite if and only if $x \in \mathbb{Q}(\beta)$. However, for a non-Pisot algebraic integer β it is possible that $\hat{J}_\beta(x)$ is finite for some x, e.g., see Examples 3.2 and 3.5. But we have not found a deeper characterization of these xs.

For $\beta > 1$ and $k \in \mathbb{N}$, a function $g(y)$ is said to be (β, k)-type if there exists an eventually periodic sequence $(d_i) \in \mathbb{Z}^{\mathbb{N}}$ such that

$$g(y) = \sum_{i=1}^\infty d_i y^i \text{ with } k = \min\{i : d_i \neq 0\} \text{ and } g(\beta^{-1}) = 0.$$

Obviously, $g(y)$ is well defined for $-1 < y < 1$. Our main theorem in the present paper is the following theorem.

Theorem 1.1. *Suppose $\hat{J}_\beta(x)$ is finite. Then the following statements are equivalent*:

(i) $\#\Sigma_\beta(x) = 2^{\aleph_0}$.
(ii) $\dim_H \Sigma_\beta(x) = \lim_{n\to\infty} \frac{1}{n} \log_{\lfloor\beta\rfloor+1} \#\Sigma_{\beta,n}(x) > 0$.
(iii) *There exist* $\mathbf{a} \in \bigcup_{n\geq 0}\{0,1,\ldots,\lfloor\beta\rfloor\}^n$ *and* $\mathbf{b},\mathbf{c} \in \bigcup_{n\geq 1}\{0,1,\ldots,\lfloor\beta\rfloor\}^n$ *with* $\mathbf{i}(\mathbf{b}) \neq \mathbf{i}(\mathbf{c})$ *such that* $\mathbf{ab}^\infty, \mathbf{ac}^\infty \in \Sigma_\beta(x)$.
(iv) *There exist an eventually periodic sequence* $(\alpha_j) = \delta_1 \ldots \delta_k(\delta_{k+1}\ldots\delta_{k+\ell})^\infty \in \Sigma_\beta(x)$ *with* $k \geq 0$, $\ell \geq 1$ *and a* $(\beta, k+1)$*-type function* $g(y) = \sum_{i=1}^\infty d_i y^i$ *such that* $(\alpha_i - d_i)_{i=1}^\infty \in \{0,1,\ldots,\lfloor\beta\rfloor\}^{\mathbb{N}}$ *is of form* $(\alpha_i - d_i)_{i=1}^\infty = (\alpha_i - d_i)_{i=1}^m (\delta_{k+1}\ldots\delta_{k+\ell})^\infty$ *with* $m \geq k+1$.

This paper is arranged as follows. A graph-directed construction will be described in Sec. 2. The final section is devoted to the proof of Theorem 1.1.

2. Graph-Directed Construction

We make a graph-directed construction $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to describe the set $\Sigma_\beta(x)$. Let $\hat{J}_\beta(x) = \{v_i\}_{i=1}^k$ with $v_1 = x$. We take $\{v_i\}_{i=1}^k$ as the vertex set $\mathcal{V}$. For each vertex pair $v_i, v_j \in \mathcal{V}$ we say $e \in \{0,1,\ldots,\lfloor\beta\rfloor\}$ a directed edge starting at v_i and terminating at v_j if $T_{\beta,e}(v_i) = v_j$. Thus each vertex pair v_i, v_j has at most one directed edge starting at v_i and terminating at v_j. And for each vertex v_i there exist at least one and at most two directed edges starting at v_i, the later occurs if and only if $v_i \in \hat{S}_\beta(x)$. The directed edge set $\mathcal{E}$ consists of all such possible directed edges e. For a directed edge $e \in \mathcal{E}$ we use $\mathbf{i}(e)$ and $\mathbf{t}(e)$ to denote its starting and terminating vertices, respectively. Note that an edge e indeed is a triple $(\mathbf{i}(e), e, \mathbf{t}(e))$. So it is possible that a digit e may occur in $\mathcal{E}$ many times which stands for distinct edges.

An edge e with $\mathbf{i}(e) = \mathbf{t}(e)$ is called a *self-loop*. A finite *path* on the graph $\mathcal{G}$ is a finite sequence $e_1e_2\ldots e_n$ of edges from $\mathcal{E}$ such that $\mathbf{t}(e_j) = \mathbf{i}(e_{j+1})$ for all $1 \leq j \leq n-1$. An infinite *path* on the graph $\mathcal{G}$ is an infinite sequence $e_1e_2\ldots$ of edges from $\mathcal{E}$ such that $\mathbf{t}(e_j) = \mathbf{i}(e_{j+1})$ for all $j \geq 1$. A *cycle* is a finite path that starts and terminates at the same vertex. The graph $\mathcal{G}$ is called *strongly connected* if for each pair of vertices v_i and v_j there is a finite path starting at v_i and terminating at v_j.

From the construction of $\mathcal{G}$ it follows that each β-expansion of x can be identified with an infinite path starting at v_1 on $\mathcal{G}$, i.e.

$$\Sigma_\beta(x) = \{e_1e_2e_3\ldots : | \, e_i \in \mathcal{E}, \mathbf{i}(e_1) = v_1 \text{ and } \mathbf{t}(e_j) = \mathbf{i}(e_{j+1}) \text{ for all } j \geq 1\}.$$

So $\Sigma_\beta(x)$ is singleton if and only if $\hat{S}_\beta(x) = \emptyset$.

The incidence matrix $A = (a_{i,j})_{k\times k}$ of $\mathcal{G}$ is a $0-1$ matrix such that $a_{i,j} = 1$ if and only if there exists an $e \in \mathcal{E}$ with $\mathbf{i}(e) = v_i$ and $\mathbf{t}(e) = v_j$.

We endow $\{0,1,\ldots,\lfloor\beta\rfloor\}^{\mathbb{N}}$ with the metric $d(\cdot,\cdot)$: for $\varepsilon=(\varepsilon_i),\delta=(\delta_i)\in\{0,1,\ldots,\lfloor\beta\rfloor\}^{\mathbb{N}}$

$$d(\varepsilon,\delta)=\begin{cases}(\lfloor\beta\rfloor+1)^{-n} & \text{if } \varepsilon\neq\delta \text{ and } n=\min\{i:\delta_i\neq\varepsilon_i\},\\ 0 & \text{if } \varepsilon=\delta.\end{cases}$$

The following lemma essentially comes from [3, Theorem 5.2].

Lemma 2.1. *Suppose that $\hat{J}_\beta(x)$ is finite. If the graph $\mathcal{G}$ associated to $\hat{J}_\beta(x)$ is strongly connected, then*

$$\dim_H \Sigma_\beta(x)=\lim_{n\to\infty}\frac{\log_{\lfloor\beta\rfloor+1}\#\Sigma_{\beta,n}(x)}{n}=\log_{\lfloor\beta\rfloor+1}\lambda.$$

where λ is the spectral radius of the incidence matrix $A=(a_{i,j})_{k\times k}$ of $\mathcal{G}$.

In fact, according to Perron–Frobenius Theorem λ is an eigenvalue of A which has a right eigenvector w whose components are all positive. Let $w=(w_1,\ldots,w_k)^T$. Then for each $v_i\in\hat{J}_\beta(x)$, we have

$$\frac{\min_i w_i}{\max_i w_i}\lambda^n\leq\#\Sigma_{\beta,n}(v_i)\leq\frac{\max_i w_i}{\min_i w_i}\lambda^n\quad\text{for each } n\in\mathbb{N}.$$

The desired results can be obtained by a verbatim textual argument in [3].

A *subgraph* of $\mathcal{G}=(\mathcal{V},\mathcal{E})$ is a tuple $(\mathcal{V}^*,\mathcal{E}^*)$ where $\mathcal{V}^*\subseteq\mathcal{V}$ and $\mathcal{E}^*=\{e\in\mathcal{E}:\mathbf{i}(e),\mathbf{t}(e)\in\mathcal{V}^*\}$. We say a subgraph $(\mathcal{V}^*,\mathcal{E}^*)$ of $\mathcal{G}$ is maximal strongly connected if it is strongly connected and no strongly connected subgraph $(\mathcal{V}^{**},\mathcal{E}^{**})$ such that $\mathcal{V}^{**}\supsetneqq\mathcal{V}^*$. Let $\tilde{\mathcal{G}}$ be the collection of all maximal strongly connected subgraphs of $\mathcal{G}$. We have $\tilde{\mathcal{G}}\neq\emptyset$ by the finiteness of $\hat{J}_\beta(x)$.

For a $\mathcal{H}\in\tilde{\mathcal{G}}$ we denote by $A_{\mathcal{H}}$ the incidence matrix and by $\lambda_{\mathcal{H}}$ the spectral radius of $A_{\mathcal{H}}$.

Proposition 2.2. *Suppose that $\hat{J}_\beta(x)$ is finite and that $\mathcal{G}$ is the graph associated to $\hat{J}_\beta(x)$. Then*

$$\dim_H \Sigma_\beta(x)=\lim_{n\to\infty}\frac{\log_{\lfloor\beta\rfloor+1}\#\Sigma_{\beta,n}(x)}{n}=\log_{\lfloor\beta\rfloor+1}\lambda.$$

where λ is the spectral radius of the incidence matrix $A=(a_{i,j})_{k\times k}$ of $\mathcal{G}$.

Proof. First we have

$$\dim_H \Sigma_\beta(x)\leq\lim_{n\to\infty}\frac{\log_{\lfloor\beta\rfloor+1}\#\Sigma_{\beta,n}(x)}{n}=\log_{\lfloor\beta\rfloor+1}\lambda.$$

The former inequality can be verified directly and the later equality is from [18, Theorem 4.4.4].

Now we take $\mathcal{H}=(\mathcal{V}^*,\mathcal{E}^*)\in\tilde{\mathcal{G}}$ such that $\lambda_{\mathcal{H}}=\lambda$. Then $\dim_H\Sigma_\beta(x)\geq\dim_H\Sigma_\beta(y)$ with $y\in\mathcal{V}^*$. However we have $\dim_H\Sigma_\beta(y)=\log_{\lfloor\beta\rfloor+1}\lambda_{\mathcal{H}}$ by Lemma 2.1. $\square$

The following theorem shows the relation between cardinality of $\Sigma_\beta(x)$ and the graph $\mathcal{G}$.

Theorem 2.3. *Suppose that $\hat{J}_\beta(x)$ is finite. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph associated to $\hat{J}_\beta(x)$ and let $\tilde{\mathcal{G}}$ be the collection of all maximal strongly connected subgraphs of $\mathcal{G}$. Then*

(a) $\#\Sigma_\beta(x) = 2^{\aleph_0}$ *if and only if there exist a $(\mathcal{V}^*, \mathcal{E}^*) \in \tilde{\mathcal{G}}$ and a $v \in \mathcal{V}^*$ such that* $\#\{e \in \mathcal{E}^* : \mathbf{i}(e) = v\} = 2$;
(b) $\#\Sigma_\beta(x) = \aleph_0$ *if and only if for each $(\mathcal{V}^*, \mathcal{E}^*) \in \tilde{\mathcal{G}}$ we have $\#\{e \in \mathcal{E}^* : \mathbf{i}(e) = v\} = 1$ for all $v \in \mathcal{V}^*$, and there exists a $(\mathcal{V}^{**}, \mathcal{E}^{**}) \in \tilde{\mathcal{G}}$ such that $\mathcal{V}^{**} \cap \hat{S}_\beta(x) \neq \emptyset$;*
(c) $\#\Sigma_\beta(x) < \aleph_0$ *if and only if for each $(\mathcal{V}^*, \mathcal{E}^*) \in \tilde{\mathcal{G}}$ we have $\mathcal{V}^* \cap \hat{S}_\beta(x) = \emptyset$.*

Proof. It suffices to prove the sufficiency parts.

(a) Suppose that there exist a $(\mathcal{V}^*, \mathcal{E}^*) \in \tilde{\mathcal{G}}$ and a $v \in \mathcal{V}^*$ such that $\#\{e \in \mathcal{E}^* : \mathbf{i}(e) = v\} = 2$. Let e_1, e_2 be distinct edges from $\mathcal{E}^*$ both of which start at v. Let $a_1 \cdots a_k$ be a path on $(\mathcal{V}^*, \mathcal{E}^*)$ which connects the vertices $\mathbf{t}(e_1)$ and v. Let $b_1 \cdots b_\ell$ be a path on $(\mathcal{V}^*, \mathcal{E}^*)$ which connects the vertices $\mathbf{t}(e_2)$ and v. Then

$$(e_1 a_1 \cdots a_k)^\infty, (e_2 b_1 \cdots b_\ell)^\infty \in \Pi^{-1}(v). \tag{2.1}$$

Set $\mathbf{a} = e_1 a_1 \cdots a_k$ and $\mathbf{b} = e_2 b_1 \cdots b_\ell$. Thus $\{\mathbf{a}, \mathbf{b}\}^{\mathbb{N}} \subseteq \Pi^{-1}(v)$ which implies that $\#\Sigma_\beta(v) = 2^{\aleph_0}$ and so $\#\Sigma_\beta(x) = 2^{\aleph_0}$.

(b) Let $v \in \mathcal{V}^{**} \cap \hat{S}_\beta(x)$. Let both edges e_1 and e_2 start at v such that $\mathbf{t}(e_1) \in \mathcal{V}^{**}$ and $\mathbf{t}(e_2) \notin \mathcal{V}^{**}$. Denote $e_1 a_1 \cdots a_k$ is a cycle on $(\mathcal{V}^{**}, \mathcal{E}^{**})$, i.e. $\mathbf{t}(a_k) = v$. If $\#\mathcal{V}^{**} = 1$, then this cycle is just e_1 with $\mathbf{i}(e_1) = \mathbf{t}(e_1) = v$. Then for any $(\delta_i) \in \Pi^{-1}(\mathbf{t}(e_2))$

$$\Pi^{-1}(v) \supseteq \{(e_1 a_1 \cdots a_k)^\ell e_2 * (\delta_i) : \ell \in \mathbb{N}\},$$

which implies that $\#\Pi^{-1}(v) \geq \aleph_0$ and so $\#\Sigma_\beta(x) \geq \aleph_0$.

Denote $\mathcal{T} = \mathcal{E} \backslash \bigcup_{(\mathcal{V}^*, \mathcal{E}^*) \in \tilde{\mathcal{G}}} \mathcal{E}^*$. Now for a $(\delta_i) \in \Sigma_\beta(x)$, By the assumption we know that each edge from $\mathcal{T}$ may appear in (δ_i) at most one time, and that when edges e_1, e_2 from some $\mathcal{E}^*$ appear in (δ_i), say $\delta_k = e_1$ and $\delta_{k+\ell} = e_2$, then the block $\delta_k \cdots \delta_{k+\ell}$ is a path in $(\mathcal{V}^*, \mathcal{E}^*)$ and is uniquely determined by e_1, e_2 and ℓ. Therefore $\#\Sigma_\beta(x) \leq \aleph_0$.

(c) From the assumption it follows that any infinite path surely goes in some subgraph and then never goes out. Thus $\Sigma_\beta(x)$ is just a finite set. □

From the proof of Theorem 2.3(a) we have the following corollary even without the finiteness of $\hat{J}_\beta(x)$.

Corollary 2.4. *If there exist $\mathbf{a}, \mathbf{b} \in \bigcup_{k \geq 1} \{0, 1, \ldots, \lfloor \beta \rfloor\}^k$ such that $\mathbf{i}(\mathbf{a}) \neq \mathbf{i}(\mathbf{b})$ and $\Pi(\mathbf{a}^\infty) = \Pi(\mathbf{b}^\infty) = v \in \hat{J}_\beta(x)$, then $\#\Sigma_\beta(x) = 2^{\aleph_0}$.*

3. Proof of Theorem 1.1 and Examples

In this section we give the proof of Theorem 1.1 and some examples.

Proof of Theorem 1.1. First we have that $\dim_H \Sigma_\beta(x) = \lim_{n\to\infty} \frac{1}{n} \log_{\lfloor\beta\rfloor+1} \# \Sigma_{\beta,n}(x)$ by Proposition 2.2.

(i) $\Rightarrow$ (ii) By Theorem 2.3(a) one can choose $\mathcal{H} = (\mathcal{V}^*, \mathcal{E}^*) \in \tilde{\mathcal{G}}$ with $\#\{e \in \mathcal{E}^* : \mathbf{i}(e) = v\} = 2$ for some $v \in \mathcal{V}^*$. Thus its incidence matrix $A_{\mathcal{H}}$ has spectral radius $\lambda_{\mathcal{H}} > 1$. Hence $\dim_H \Sigma_\beta(x) > 0$ by Proposition 2.2.

(ii) $\Rightarrow$ (iii) For this case one has $\#\Sigma_\beta(x) = 2^{\aleph_0}$ and so by Theorem 2.3(a) one can obtain a $v \in \hat{J}_\beta(x)$ and $\mathbf{b}, \mathbf{c} \in \bigcup_{n\ge1}\{0,1,\ldots,\lfloor\beta\rfloor\}^n$ with $\mathbf{i}(\mathbf{b}) \ne \mathbf{i}(\mathbf{c})$ such that $\mathbf{b}^\infty, \mathbf{c}^\infty \in \Pi^{-1}(v)$ (see (2.1)), which implies (iii).

(iii) $\Rightarrow$ (iv) Denote $(\alpha_i) = \mathbf{ab}^\infty = \delta_1 \ldots \delta_k(\delta_{k+1}\ldots\delta_{k+\ell})^\infty$. Let $(d_i) = \mathbf{ab}^\infty - \mathbf{acb}^\infty$. Here $(\gamma_i) - (\varepsilon_i)$ means that $(\gamma_i - \varepsilon_i)$. Then $\min\{i : d_i \ne 0\} = k+1$ and $(d_i) \in \mathbb{Z}^{\mathbb{N}}$ is an eventually periodic sequence. By letting $g(y) = \sum_{i=1}^\infty d_i y^i$ one has

$$g(\beta^{-1}) = 0 \quad \text{and} \quad (\alpha_i - d_i)_{i=1}^\infty = \mathbf{acb}^\infty \in \{0,1,\ldots,\lfloor\beta\rfloor\}^{\mathbb{N}},$$

where the first equality is obtained by the fact that $\mathbf{ab}^\infty, \mathbf{acb}^\infty \in \Sigma_\beta(x)$.

(iv) $\Rightarrow$ (i) We have

$$(\alpha_i - d_i)_{i=1}^m (\delta_{k+1}\ldots\delta_{k+\ell})^\infty = \delta_1\ldots\delta_k\gamma_1\ldots\gamma_{m-k}(\delta_{k+1}\ldots\delta_{k+\ell})^\infty \in \Sigma_\beta(x).$$

Thus $\delta_1\ldots\delta_k(\gamma_1\cdots\gamma_{m-k})^\infty \in \Sigma_\beta(x)$ with $\gamma_1 \ne \delta_{k+1}$. Let $\mathbf{u} = \delta_{k+1}\ldots\delta_{k+\ell}$ and $\mathbf{v} = \gamma_1\cdots\gamma_{m-k}$. Thus $\#\Sigma_\beta(x) = 2^{\aleph_0}$ since

$$\Sigma_\beta(x) \supseteq \{(\delta_1\ldots\delta_k)(s_i)_{i=1}^\infty : (s_i)_{i=1}^\infty \in \{\mathbf{u},\mathbf{v}\}^{\mathbb{N}}\}. \qquad \square$$

From the proof of Theorem 1.1 we have the following corollary even without the finiteness of $\hat{J}_\beta(x)$.

Corollary 3.1. *We have* $\#\Sigma_\beta(x) = 2^{\aleph_0}$ *if either* (iii) *or* (iv) *in Theorem* 1.1 *holds.*

In the following we give several examples.

Example 3.2. Let $\beta \approx 1.68042$ be the positive root of equation $y^5 - y^4 - y^3 - y + 1 = 0$. Then β is not Pisot, since $y^5 - y^4 - y^3 - y + 1$ is the minimal polynomial of β and has a root in $(-\infty, -1)$. One can check that

$$\hat{J}_\beta(1) = \{1, \beta-1, \beta^2-\beta, \beta^3-\beta^2-1, \beta^4-\beta^3-\beta-1, \beta^2-\beta-1, \beta^3-\beta^2-\beta, \beta^4-\beta^3-\beta^2\}$$

and $\hat{S}_\beta(1) = \{\beta-1\}$.

More exactly we have (recall that $T_{\beta,k}(x) = \beta x - k$, $k \in \{0,1,\ldots,\lfloor\beta\rfloor\} = \{0,1\}$)

$$1 = v_1 \xrightarrow{T_{\beta,1}} v_2 = \beta-1, \qquad \beta-1 = v_2 \xrightarrow{T_{\beta,0}} v_3 = \beta^2-\beta,$$

$$\beta^2-\beta = v_3 \xrightarrow{T_{\beta,1}} v_4 = \beta^3-\beta^2-1, \quad \beta^3-\beta^2-1 = v_4 \xrightarrow{T_{\beta,1}} v_5 = \beta^4-\beta^3-\beta-1,$$

$$\beta^4-\beta^3-\beta-1 = v_5 \xrightarrow{T_{\beta,0}} v_4 = \beta^3-\beta^2-1, \qquad \beta-1 = v_2 \xrightarrow{T_{\beta,1}} v_6 = \beta^2-\beta-1,$$

$$\beta^2-\beta-1 = v_6 \xrightarrow{T_{\beta,0}} v_7 = \beta^3-\beta^2-\beta, \quad \beta^3-\beta^2-\beta = v_7 \xrightarrow{T_{\beta,0}} v_8 = \beta^4-\beta^3-\beta^2,$$

$$\beta^4-\beta^3-\beta^2, = v_8 \xrightarrow{T_{\beta,0}} v_2 = \beta-1.$$

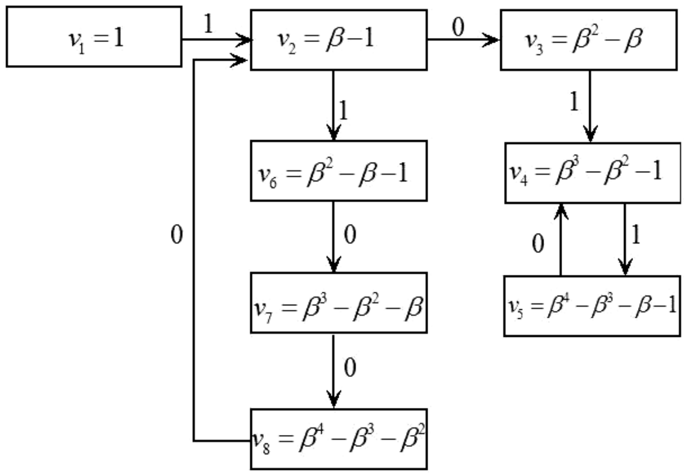

Fig. 1. The graph $\mathcal{G}$ associated to $\hat{J}_\beta(1)$. $\tilde{\mathcal{G}} = \{(\mathcal{V}_1^*, \mathcal{E}_1^*), (\mathcal{V}_2^*, \mathcal{E}_2^*)\}$ with $\mathcal{V}_1^* = \{v_4, v_5\}$ and $\mathcal{V}_2^* = \{v_2, v_6, v_7, v_8\}$, $\mathcal{V}_2^* \cap \hat{S}_\beta(1) \neq \emptyset$.

By Theorem 2.3(b) we have $\#\Sigma_\beta(1) = \aleph_0$. The graph $\mathcal{G}$ is illustrated in Fig. 1.

Example 3.3. Let $\kappa \in \mathbb{N}$ and $\epsilon \in \{1, \ldots, \kappa\}$. Let $\beta \in (\kappa, \kappa+1)$ be the positive root of the equation $y^4 - \kappa y^3 - \epsilon y^2 - y + \epsilon = 0$. Then $\#\Sigma_\beta(1) = 2^{\aleph_0}$.

Proof. We first like to point out that there exists $\beta \in (\kappa, \kappa+1)$ such that

$$\beta^4 - \kappa\beta^3 - \epsilon\beta^2 - \beta + \epsilon = 0, \text{ or equivalently } 1 - \frac{\kappa}{\beta} - \frac{\epsilon}{\beta^2} - \frac{1}{\beta^3} + \frac{\epsilon}{\beta^4} = 0. \tag{3.1}$$

In fact, by letting $f(y) = y^4 - \kappa y^3 - \epsilon y^2 - y + \epsilon$ it has that both $f(\kappa) < 0$ and $f(\kappa+1) > 0$ hold for all $\epsilon \in \{1, \ldots, \kappa\}$. One can check that

$$(1 - \kappa y - \epsilon y^2 - y^3 + \epsilon y^4) \sum_{j=0}^{\infty} y^{3j} = 1 - \sum_{j=1}^{\infty} \alpha_j y^j \text{ with } (\alpha_j)_{j=1}^{\infty} = \kappa(\epsilon 0(\kappa - \epsilon))^{\infty}.$$

From (3.1) it follows that $(\alpha_j)_{j=1}^{\infty} = \kappa(\epsilon 0(\kappa-\epsilon))^{\infty} \in \Sigma_\beta(1)$. Now let

$$\begin{aligned} g(y) &= (y^2 - y^6)(1 - \kappa y - \epsilon y^2 - y^3 + \epsilon y^4) \sum_{j=0}^{\infty} y^{3j} \\ &= (y^2 - y^6)\left(1 - \sum_{j=1}^{\infty} \alpha_j y^j\right) := \sum_{i=1}^{\infty} d_i y^i. \end{aligned}$$

Then $g(y)$ is a $(\beta, 2)$-type function and $(\alpha_i - d_i)_{i=1}^{\infty} = \kappa(\epsilon-1)\kappa\kappa\epsilon(\kappa-\epsilon+1)00(\kappa-\epsilon)(\epsilon 0(\kappa-\epsilon))^{\infty}$. It follows from Theorem 1.1(iv) (or Corollary 3.1) that $\#\Sigma_\beta(1) = 2^{\aleph_0}$. □

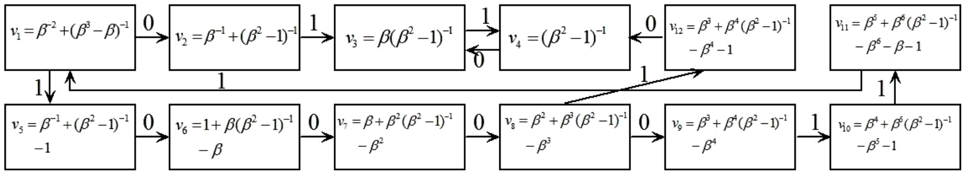

Fig. 2. The graph $\mathcal{G}$ associated to $\hat{J}_\beta(x)$. $\tilde{\mathcal{G}} = \{(\mathcal{V}_1^*, \mathcal{E}_1^*), (\mathcal{V}_2^*, \mathcal{E}_2^*)\}$ with $\mathcal{V}_1^* = \{v_3, v_4\}$ and $\mathcal{V}_2^* = \{v_1, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$, $\mathcal{V}_2^* \cap \hat{S}_\beta(x) \neq \emptyset$, $x = \beta^{-2} + (\beta^3 - \beta)^{-1}$.

Example 3.4. If the greedy and lazy β-expansions of 1 are $\varepsilon_1 \dots \varepsilon_k$ and $(\varepsilon_1 \dots \varepsilon_{k-1}(\varepsilon_k - 1))^\infty$, respectively, then $\#\Sigma_\beta(1) = \aleph_0$.

Proof. By the assumption we have $\Sigma_{\beta,k}(1) = \{(\delta_i)|k : (\delta_i) \in \Pi^{-1}(1)\} = \{\varepsilon_1 \dots \varepsilon_k, \varepsilon_1 \dots \varepsilon_{k-1}(\varepsilon_k - 1)\}$. Thus we have

$$1 = v_1 \xrightarrow{T_{\beta,\varepsilon_1}} v_2 \xrightarrow{T_{\beta,\varepsilon_2}} v_3 \cdots \xrightarrow{T_{\beta,\varepsilon_{k-1}}} v_k \xrightarrow{T_{\beta,\varepsilon_k}} v_{k+1} = 0 \xrightarrow{T_{\beta,0}} v_{k+1} \quad \text{and} \quad v_k \xrightarrow{T_{\beta,\varepsilon_k - 1}} v_1.$$

Therefore, $\#\Sigma_\beta(1) = \aleph_0$ by Theorem 2.3(b). □

Example 3.5. Let $\beta \approx 1.65462$ be the positive root of $y^6 - 2y^4 - y^3 - 1 = 0$. Then β is not Pisot, since $y^6 - 2y^4 - y^3 - 1$ is the minimal polynomial of β and $y \approx -1.26493$ is also a root. Let $x = \beta^{-2} + (\beta^3 - \beta)^{-1}$. The sets $\hat{J}_\beta(x) = \{v_i\}_{i=1}^{12}$ and $\hat{S}_\beta(x) = \{v_1, v_8\}$ are illustrated in Fig. 2. Then $\#\Sigma_\beta(x) = \aleph_0$ by Theorem 2.3(b).

Acknowledgments

The authors would like to thank the anonymous referees for their helpful comments and suggestions, which leads to a significant improvement of the manuscript. The first and third authors were supported by the National Natural Science Foundation of China (NSFC) #11201312, #61373087, #61272252, #61472257, the Foundation for Distinguished Young Teachers in Guangdong, China #Yq2013144, and Guangdong Natural Science Foundation of China #2015A030313557, #2015A030313550. The second author was supported by the National Natural Science Foundation of China (NSFC) #11271137 and Science and Technology Commission of Shanghai Municipality (STCSM), #13dz2260400.

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